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MULTIDIMENSIONAL YAMADA-WATANABE THEOREM AND ITS APPLICATIONS

PIOTR GRACZYK, JACEK MAŁECKI

ABSTRACT. Multidimensional and matrix versions of the Yamada-Watanabe theorem are proved. They are applied to particle systems of squared Bessel processes and to matrix analogues of squared Bessel processes: Wishart and Jacobi matrix processes.

1. INTRODUCTION

In this note we prove a multidimensional and matrix analogues of the celebrated Yamada-Watanabe theorem, ensuring the existence and uniqueness of strong solutions of one-dimensional stochastic differential equations (SDEs) with a Hölder coefficient in the Itô integral part. In Section 2 we prove a multidimensional Yamada-Watanabe theorem (Theorem 2). In Section 3 we show the existence and uniqueness of a strong solution of the matrix SDEs

$$dX_t = g(X_t)dB_t h(X_t) + h(X_t)dB_t^T g(X_t) + b(X_t)dt$$

where $g, h, b : \mathbf{R} \rightarrow \mathbf{R}$ are such that $g \otimes h$ is $1/2$ -Hölder continuous symmetrized $g^2 \otimes h^2$ and b are Lipschitz continuous and B_t is a Brownian $p \times p$ matrix, and establish in this way a matrix Yamada-Watanabe theorem, see Theorem 5. Section 4 contains interesting applications. We apply Theorems 2 and 5 to

(i) noncolliding particle systems of squared Bessel processes which are intensely studied in recent years in statistical and mathematical physics (Katori, Tanemura [13, 14]).

(ii) the systems of SDEs for the eigenvalues of Wishart and Jacobi matrix processes, as well as to the β -Wishart and β -Jacobi processes. We note the importance of the β -Wishart systems in statistical physics: they are statistical mechanics models of “log-gases”, see the recent book of Forrester [11].

Surprisingly, the existence of strong solutions of SDEs for such “Hölder” non-colliding particle systems was not established in general; only some cases of (ii) were treated by Demni [7, 8]. In Sections 4.1 and 4.4 we prove the existence and uniqueness of a strong solution to all these systems of SDEs.

The matrix Yamada-Watanabe theorem is applied in Sections 4.2 and 4.3 to some matrix valued squared Bessel type processes. We improve the known results of Bru [2, 3], Mayerhofer et al. [17] and Doumerc [10] on the existence and uniqueness of strong solutions to Wishart and Jacobi matrix SDEs. We extend them to the whole range of the drift parameter $b(X_t)$. In the Wishart case we contribute in this way to realization of a programme started by Donati-Martin, Doumerc, Matsumoto, Yor [9], claiming that Wishart processes have similar properties as classical 1-dimensional squared Bessel processes.

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2. A MULTIDIMENSIONAL YAMADA-WATANABE THEOREM

Let us recall the classical Yamada-Watanabe theorem, see e.g. [12], p.168 and [21].

Theorem 1. *Let $B(t)$ be a Brownian motion on \mathbf{R} . Consider the SDE*

$$dX(t) = \sigma(X(t))dB(t) + b(X(t))dt.$$

If $|\sigma(x) - \sigma(y)|^2 \leq \rho(|x - y|)$ for a strictly increasing function ρ on \mathbf{R}^+ with $\rho(0) = 0$ and $\int_{0+} \rho^{-1}(x)dx = \infty$, and b is Lipschitz continuous, then the pathwise uniqueness of solutions holds; consequently the equation has a unique strong solution.

No multidimensional versions of the Yamada-Watanabe theorem seem to be known, even if their need is great (cf. [3], p. 738). We propose a useful generalization, however we stress the fact that the Hölder continuous functions σ_i appearing in the following system of SDEs are one-dimensional. The proof is based on the approach presented in Revuz, Yor [20], in particular on the results of Le Gall [16]. By $\|\cdot\|$ we mean the Euclidean norm $\|\cdot\|_2$ on \mathbf{R}^d .

Theorem 2. *Let $p, q, r \in \mathbb{N}$ and the functions $b_i : \mathbf{R}^p \rightarrow \mathbf{R}$, $i = 1, \dots, p$ and $c_k, d_j : \mathbf{R}^{p+r} \rightarrow \mathbf{R}$, $k = p+1, \dots, p+q$, $j = p+1, \dots, p+r$, be bounded real-valued and continuous, satisfying the following Lipschitz conditions*

$$\begin{aligned} |b_i(y_1) - b_i(y_2)| &\leq A\|y_1 - y_2\|, \quad i = 1, \dots, p, \\ |c_k(y_1, z_1) - c_k(y_2, z_2)| &\leq A\|(y_1, z_1) - (y_2, z_2)\|, \quad k = p+1, \dots, p+q, \\ |d_j(y_1, z_1) - d_j(y_2, z_2)| &\leq A\|(y_1, z_1) - (y_2, z_2)\|, \quad j = p+1, \dots, p+r, \end{aligned}$$

for every $y_1, y_2 \in \mathbf{R}^p$ and $z_1, z_2 \in \mathbf{R}^r$. Moreover, let $\sigma_i : \mathbf{R} \rightarrow \mathbf{R}$, $i = 1, \dots, p$, be a set of bounded Borel functions such that

$$|\sigma_i(x) - \sigma_i(y)|^2 \leq \rho_i(|x - y|), \quad x, y \in \mathbf{R},$$

where $\rho_i : (0, \infty) \rightarrow (0, \infty)$ are Borel functions such that $\int_{0+} \rho_i^{-1}(x)dx = \infty$. Then the pathwise uniqueness holds for the following system of stochastic differential equations

$$dY_i = \sigma_i(Y_i)dB_i + b_i(Y)dt, \quad i = 1, \dots, p, \quad (2.1)$$

$$dZ_j = \sum_{k=p+1}^{p+q} c_k(Y, Z)dB_k + d_j(Y, Z)dt, \quad j = p+1, \dots, p+r. \quad (2.2)$$

Proof. Let (Y, Z) and (\tilde{Y}, \tilde{Z}) be two solutions with respect to the same Brownian motion $B = (B_i)_{i \leq p+q}$ such that $Y(0) = \tilde{Y}(0)$ and $Z(0) = \tilde{Z}(0)$ a.s. For every $i = 1, \dots, p$ we have

$$Y_i(t) - \tilde{Y}_i(t) = \int_0^t (\sigma_i(Y_i(s)) - \sigma_i(\tilde{Y}_i(s)))dB_i(s) + \int_0^t (b_i(Y(s)) - b_i(\tilde{Y}(s)))ds. \quad (2.3)$$

Then we get

$$\int_0^t \frac{\mathbf{1}_{\{Y_i(s) > \tilde{Y}_i(s)\}}}{\rho_i(Y_i(s) - \tilde{Y}_i(s))} d\langle Y_i - \tilde{Y}_i, Y_i - \tilde{Y}_i \rangle = \int_0^t \frac{(\sigma_i(Y_i(s)) - \sigma_i(\tilde{Y}_i(s)))^2}{\rho_i(Y_i(s) - \tilde{Y}_i(s))} \mathbf{1}_{\{Y_i(s) > \tilde{Y}_i(s)\}} ds \leq t.$$

Thus, applying Lemma 3.3 from [20] p. 389, we get that the local time of $Y_i - \tilde{Y}_i$ at 0 vanishes identically. Consequently, by Tanaka's formula we get

$$\begin{aligned} |Y_i(t) - \tilde{Y}_i(t)| &= \int_0^t \operatorname{sgn}(Y_i(s) - \tilde{Y}_i(s))d(Y_i(s) - \tilde{Y}_i(s)) + \frac{1}{2}L_t^0(Y_i - \tilde{Y}_i) \\ &= \int_0^t \operatorname{sgn}(Y_i(s) - \tilde{Y}_i(s))d(Y_i(s) - \tilde{Y}_i(s)). \end{aligned}$$

Since σ_i is bounded, using (2.3), we state that

$$|Y_i(t) - \tilde{Y}_i(t)| - \int_0^t \text{sgn}(Y_i(s) - \tilde{Y}_i(s))(b_i(Y(s)) - b_i(\tilde{Y}(s)))ds$$

is a martingale vanishing at 0. This together with the Lipschitz conditions satisfied by b_i give

$$\mathbf{E}|Y_i(t) - \tilde{Y}_i(t)| \leq A \int_0^t \mathbf{E}\|Y(s) - \tilde{Y}(s)\|ds.$$

Summing up the above-given inequalities we arrive at

$$\mathbf{E}\|Y(t) - \tilde{Y}(t)\| \leq C \int_0^t \mathbf{E}\|Y(s) - \tilde{Y}(s)\|ds$$

and Gronwall's lemma shows that $Y(t) = \tilde{Y}(t)$ for every $t > 0$ a.s.

Using in a standard way the properties of the Itô integral and the Schwarz inequality, similarly as in [12], p. 165, we get that for every $t \in [0, T]$

$$\begin{aligned} \mathbf{E}|Z_j(t) - \tilde{Z}_j(t)|^2 &\leq C \sum_{k=p+1}^{p+q} \mathbf{E} \left(\int_0^t c_k(Y(s), Z(s)) - c_k(\tilde{Y}(s), \tilde{Z}(s)) dB_k(s) \right)^2 \\ &\quad + C \mathbf{E} \left(\int_0^t d_j(Y(s), Z(s)) - d_j(\tilde{Y}(s), \tilde{Z}(s)) ds \right)^2 \\ &\leq C \sum_{k=p+1}^{p+q} \mathbf{E} \int_0^t (c_k(Y(s), Z(s)) - c_k(\tilde{Y}(s), \tilde{Z}(s)))^2 ds \\ &\quad + CT \mathbf{E} \int_0^t (d_j(Y(s), Z(s)) - d_j(\tilde{Y}(s), \tilde{Z}(s)))^2 ds \\ &\leq CA^2(q+T) \mathbf{E} \int_0^t (\|Y(s) - \tilde{Y}(s)\|^2 + \|Z(s) - \tilde{Z}(s)\|^2) ds \end{aligned}$$

Thus, using the previously proved fact that $Y = \tilde{Y}$ a.s. we get that

$$\mathbf{E}\|Z(t) - \tilde{Z}(t)\|^2 \leq CA^2(q+T)r \int_0^t \mathbf{E}\|Z(s) - \tilde{Z}(s)\|^2 ds.$$

One more application of the Gronwall's lemma ends the proof. \square

3. MATRIX STOCHASTIC DIFFERENTIAL EQUATIONS

Consider the space \mathcal{S}_p of symmetric $p \times p$ real matrices. Recall that if $g : \mathbf{R} \rightarrow \mathbf{R}$ and $X \in \mathcal{S}_p$ then $g(X) = Hg(\Lambda)H^T$, where $X = H\Lambda H^T$ is a diagonalization of X , with H an orthonormal matrix and Λ a diagonal one. Denote by B_t a Brownian $p \times p$ matrix. Let X_t be a stochastic process with values in \mathcal{S}_p such that $X_0 \in \tilde{\mathcal{S}}_p$, the set of symmetric matrices with p different eigenvalues. Let $\Lambda_t = \text{diag}(\lambda_i(t))$ be the diagonal matrix of eigenvalues of X_t ordered increasingly: $\lambda_1(t) \leq \lambda_2(t) \leq \dots \leq \lambda_p(t)$ and H_t an orthonormal matrix of eigenvectors of X_t . Matrices Λ and H may be chosen as smooth functions of X until the first collision time $\tau = \inf\{t : \lambda_i(t) = \lambda_j(t) \text{ for some } i \neq j\}$, cf. [19].

3.1. Eigenvalues and eigenvectors of X_t . In order to prove a matrix Yamada-Watanabe theorem we need to consider the SDEs satisfied by the processes of eigenvalues and eigenvectors of X_t .

Theorem 3. Suppose that an \mathcal{S}_p -valued stochastic process X_t satisfies the following matrix stochastic differential equation

$$dX_t = g(X_t)dB_t h(X_t) + h(X_t)dB_t^T g(X_t) + b(X_t)dt \quad (3.1)$$

where $g, h, b : \mathbf{R} \rightarrow \mathbf{R}$, and $X_0 \in \tilde{\mathcal{S}}_p$.

Let $G(x, y) = g^2(x)h^2(y) + g^2(y)h^2(x)$. Then, for $t < \tau$ the eigenvalues process Λ_t and the eigenvectors process H_t verify the following stochastic differential equations

$$d\lambda_i = 2g(\lambda_i)h(\lambda_i)d\nu_i + \left(b(\lambda_i) + \sum_{k \neq i} \frac{G(\lambda_i, \lambda_k)}{\lambda_i - \lambda_k} \right) dt \quad (3.2)$$

$$dh_{ij} = \sum_{k \neq j} h_{ik} \frac{\sqrt{G(\lambda_j, \lambda_k)}}{\lambda_j - \lambda_k} d\beta_{kj} - \frac{1}{2} h_{ij} \sum_{k \neq j} \frac{G(\lambda_j, \lambda_k)}{(\lambda_k - \lambda_j)^2} dt \quad (3.3)$$

where $(\nu_i)_i$ and $(\beta_{kj})_{k < j}$ are independent Brownian motions and $\beta_{jk} = \beta_{kj}$.

Proof. The proof generalizes ideas of Bru [1] in the case of Wishart processes. Following [10] in the case of matrix Jacobi processes, it is handy to use the Stratonovich differential notation $X \circ dY = X dY + \frac{1}{2} dX dY$. We then write the Itô product formula

$$d(XY) = dX \circ Y + X \circ dY.$$

We also have $dX \circ (YZ) = (dX \circ Y) \circ Z$ and $(X \circ dY)^T = dY^T \circ X^T$.

Define A , a stochastic logarithm of H , by

$$dA = H^{-1} \circ dH = H^T \circ dH.$$

Observe that by Itô formula applied to $H^T H = I$, the matrix A is skew-symmetric. By Itô formula applied to $\Lambda = H^T X H$, setting $dN = H^T \circ dX \circ H$, we get

$$d\Lambda = dN + \Lambda \circ dA - dA \circ \Lambda.$$

The process $\Lambda \circ dA - dA \circ \Lambda$ is zero on the diagonal. Consequently $d\lambda_i = dN_{ii}$ and $0 = dN_{ij} + (\lambda_i - \lambda_j) \circ dA_{ij}$, when $i \neq j$. Thus

$$dA_{ij} = \frac{1}{\lambda_j - \lambda_i} \circ dN_{ij}, \quad i \neq j. \quad (3.4)$$

For further computations we need the quadratic variation $\langle X_{st}, X_{s't'} \rangle$ which is easily computed from (3.1):

$$dX_{st} dX_{s't'} = g^2(X)_{ss'} h^2(X)_{tt'} + g^2(X)_{st'} h^2(X)_{s't} + g^2(X)_{s't} h^2(X)_{st'} + g^2(X)_{tt'} h^2(X)_{ss'}.$$

The martingale part of dN equals the martingale part of $H^T dX H$ and by the last formula

$$dN_{ij} dN_{km} = g^2(\Lambda)_{ik} h^2(\Lambda)_{jm} + g^2(\Lambda)_{im} h^2(\Lambda)_{jk} + g^2(\Lambda)_{jk} h^2(\Lambda)_{im} + g^2(\Lambda)_{jm} h^2(\Lambda)_{ik}. \quad (3.5)$$

From (3.5) it follows that

$$dN_{ii} dN_{jj} = 4\delta_{ij} g^2(\lambda_i) h^2(\lambda_i) dt. \quad (3.6)$$

Now we compute the finite variation part dF of dN

$$\begin{aligned} dF &= H^T b(X) H + \frac{1}{2} (dH^T dX H + H^T dX dH) \\ &= b(\Lambda) + \frac{1}{2} ((dH^T H)(H^T dX H) + (H^T dX H)(H^T dH)) \\ &= b(\Lambda) + \frac{1}{2} (dN dA + (dN dA)^T). \end{aligned}$$

Using (3.4) and (3.5) we find, writing $G(x, y) = g^2(x)h^2(y) + g^2(y)h^2(x)$,

$$(dNdA)_{ij} = \sum_{k \neq j} dN_{ik}dA_{kj} = \delta_{ij} \sum_{k \neq i} \frac{G(\lambda_i, \lambda_k)}{\lambda_i - \lambda_k}.$$

It follows that the matrix $dNdA$ is diagonal, so also dF is diagonal,

$$dF_{ii} = b(\lambda_i)dt + \sum_{k \neq i} \frac{G(\lambda_i, \lambda_k)}{\lambda_i - \lambda_k} dt$$

and by (3.4), A is a martingale. Finally, using (3.6) and the last formula, there exist independent Brownian motions ν_i , $i = 1, \dots, m$, such that (3.2) holds.

In order to find SDEs for H_t , we deduce from the definition of dA that

$$dH = H \circ dA = HdA + \frac{1}{2}dHdA = HdA + \frac{1}{2}HdAdA.$$

By (3.5) we find $dN_{ij}dN_{ij} = g^2(\lambda_i)h^2(\lambda_j) + g^2(\lambda_j)h^2(\lambda_i)$ and $dN_{ij}dN_{km} = 0$ when the ordered pairs $i < j$ and $k < m$ are different. We infer from (3.4) that

$$dA_{ij} = \frac{\sqrt{G(\lambda_i, \lambda_j)}}{\lambda_j - \lambda_i} d\beta_{ij}, \quad (3.7)$$

where the Brownian motions $(\beta_{ij})_{i < j}$ are independent and $\beta_{ji} = \beta_{ij}$. Moreover, when $k < m$, we have $d\lambda_i da_{km} = dN_{ii}dN_{km}/(\lambda_m - \lambda_k) = 0$ by (3.5), so the Brownian motions $(\beta_{ij})_{i < j}$ and $(\nu_i)_i$ are independent. From (3.7) we deduce that the matrix $dAdA$ is diagonal and

$$(dAdA)_{ii} = - \sum_{k \neq i} dA_{ik}dA_{ik} = - \sum_{k \neq i} \frac{G(\lambda_i, \lambda_k)}{(\lambda_k - \lambda_i)^2}.$$

Now we can compute $dH = HdA + \frac{1}{2}HdAdA$ and prove (3.3). \square

3.2. Complex case. In this subsection we study the eigenvalues process for a process X_t with values in the space \mathcal{H}_p of Hermitian $p \times p$ matrices.

Theorem 4. *Let W_t be a complex $p \times p$ Brownian matrix (i.e. $W_t = B_t^1 + iB_t^2$ where B_t^1 and B_t^2 are two independent real Brownian squared matrices).*

Suppose that an \mathcal{H}_p -valued stochastic process X_t satisfies the following matrix stochastic differential equation

$$dX_t = g(X_t)dW_t h(X_t) + h(X_t)dW_t^* g(X_t) + b(X_t)dt, \quad (3.8)$$

where $g, h, b : \mathbf{R} \rightarrow \mathbf{R}$, and $X_0 \in \tilde{\mathcal{H}}_p$.

Let $G(x, y) = g^2(x)h^2(y) + g^2(y)h^2(x)$. Then, for $t < \tau$ the eigenvalues process Λ_t verifies the following system of stochastic differential equations

$$d\lambda_i = 2g(\lambda_i)h(\lambda_i)d\nu_i + \left(b(\lambda_i) + 2 \sum_{k \neq i} \frac{G(\lambda_i, \lambda_k)}{\lambda_i - \lambda_k} \right) dt, \quad (3.9)$$

where $(\nu_i)_i$ are independent Brownian motions.

Proof. We will need the following formula for the quadratic variation $\langle X_{st}, X_{s't'} \rangle$ which is computed from (3.8), using the fact that for a complex Brownian motion W_t , the quadratic variation processes satisfy $d\langle W, W \rangle = 0$ and $d\langle W, \bar{W} \rangle = 2dt$.

$$dX_{st}dX_{s't'} = 2 \left(g^2(X)_{st'} h^2(X)_{s't} + g^2(X)_{s't} h^2(X)_{st'} \right). \quad (3.10)$$

Define A , a stochastic logarithm of H , by

$$dA = H^{-1} \circ dH = H^* \circ dH.$$

By Itô formula applied to $H^*H = I$, the matrix A is skew-Hermitian. In particular, the terms of $\text{diag}(A)$ are **purely imaginary** (recall that in the real case they were 0). By Itô formula applied to $\Lambda = H^*XH$, we get, setting $dN = H^* \circ dX \circ H$

$$d\Lambda = dN + \Lambda \circ dA - dA \circ \Lambda.$$

We have

$$dN = H^*dXH + \frac{1}{2}(dH^*dXH + H^*dXdH)$$

so the process N takes values in \mathcal{H}_p . In particular its diagonal entries are real. The process $\Lambda \circ dA - dA \circ \Lambda$ is zero on the diagonal, so $d\lambda_i = dN_{ii}$. Moreover, when $i \neq j$, we have $0 = dN_{ij} + (\lambda_i - \lambda_j) \circ dA_{ij}$ and

$$dA_{ij} = \frac{1}{\lambda_j - \lambda_i} \circ dN_{ij}, \quad i \neq j. \quad (3.11)$$

The martingale part of dN equals the martingale part of H^*dXH and by formula (3.10) we obtain

$$dN_{ij}dN_{km} = 2(g^2(\Lambda)_{im}h^2(\Lambda)_{jk} + g^2(\Lambda)_{jk}h^2(\Lambda)_{im}). \quad (3.12)$$

From (3.12) it follows that

$$dN_{ii}dN_{jj} = 4\delta_{ij}g^2(\lambda_i)h^2(\lambda_i)dt. \quad (3.13)$$

Now we compute the finite variation part dF of dN

$$\begin{aligned} dF &= H^*b(X)H + \frac{1}{2}(dH^*dXH + H^*dXdH) \\ &= b(\Lambda) + \frac{1}{2}((dH^*H)(H^*dXH) + (H^*dXH)(H^*dH)) \\ &= b(\Lambda) + \frac{1}{2}(dNdA + (dNdA)^*). \end{aligned}$$

Recall that $G(x, y) = g^2(x)h^2(y) + g^2(y)h^2(x)$. We get

$$(dNdA)_{ij} = \sum_k dN_{ik}dA_{kj} = 2\delta_{ij} \sum_{k \neq i} \frac{G(\lambda_i, \lambda_k)}{\lambda_i - \lambda_k} + dN_{ij}dA_{jj}.$$

When $i = j$, the term dN_{ii} is real and $dA_{ii} \in i\mathbf{R}$. It follows that

$$dF_{ii} = b(\lambda_i)dt + 2 \sum_{k \neq i} \frac{G(\lambda_i, \lambda_k)}{\lambda_i - \lambda_k} dt.$$

Finally, using (3.13) and the last formula, there exist independent Brownian motions ν_i , $i = 1, \dots, m$, such that (3.9) holds. \square

The Theorem 4 may be applied in a special case $g(x) = \sqrt{x}$, $h(x) = 1$ and $b(x) = 2\delta > 0$, when the equation (3.8) is the SDE for the complex Wishart process, called also a Laguerre process. This process and its eigenvalues were studied by König-O'Connell [15] and Demni [5].

3.3. Collision time. In this subsection we show that under some mild conditions on the functions g and h in the SDE (3.1), the eigenvalues of the process X_t never collide.

Proposition 1. *Let $\Lambda = (\lambda_i)_{i=1 \dots p}$ be a process starting from $\lambda_1(0) < \dots < \lambda_p(0)$ and satisfying (3.2) with functions $b, g, h : \mathbf{R} \rightarrow \mathbf{R}$ such that b, g^2, h^2 are Lipschitz continuous and g^2h^2 is convex or in class $\mathcal{C}^{1,1}$. Then the first collision time τ is infinite a.s.*

Proof. We define $U = -\sum_{i<j} \log(\lambda_j - \lambda_i)$ on $t \in [0, \tau]$. Applying Itô formula, using (3.2) and the fact that $d\langle \lambda_i, \lambda_j \rangle = \delta_{ij} 4g^2(\lambda_i)h^2(\lambda_j)dt$ we obtain

$$dU = \sum_{i<j} \frac{d\lambda_i - d\lambda_j}{\lambda_j - \lambda_i} + \frac{1}{2} \frac{d\langle \lambda_i, \lambda_i \rangle + d\langle \lambda_j, \lambda_j \rangle}{(\lambda_j - \lambda_i)^2} = dM + dA^{(1)} + dA^{(2)} + dA^{(3)},$$

where

$$\begin{aligned} dM &= 2 \sum_{i<j} \frac{g(\lambda_i)h(\lambda_i)d\nu_i - g(\lambda_j)h(\lambda_j)d\nu_j}{\lambda_j - \lambda_i}, \\ dA^{(1)} &= \sum_{i<j} \frac{b(\lambda_i) - b(\lambda_j)}{\lambda_j - \lambda_i} dt, \\ dA^{(2)} &= 2 \sum_{i<j} \frac{(g^2(\lambda_j) - g^2(\lambda_i))(h^2(\lambda_j) - h^2(\lambda_i))}{(\lambda_j - \lambda_i)^2} dt, \\ dA^{(3)} &= \sum_{i<j} \frac{1}{\lambda_j - \lambda_i} \sum_{k \neq i, k \neq j} \left(\frac{G(\lambda_i, \lambda_k)}{\lambda_i - \lambda_k} - \frac{G(\lambda_j, \lambda_k)}{\lambda_j - \lambda_k} \right) dt \\ &= \sum_{i<j<k} \frac{G(\lambda_j, \lambda_k)(\lambda_k - \lambda_j) - G(\lambda_i, \lambda_k)(\lambda_k - \lambda_i) + G(\lambda_i, \lambda_j)(\lambda_j - \lambda_i)}{(\lambda_j - \lambda_i)(\lambda_k - \lambda_i)(\lambda_k - \lambda_j)} dt. \end{aligned}$$

We will show that the finite variation part of U is bounded on any interval $[0, t]$. Lipschitz continuity of b , g^2 and h^2 implies that $A_t^{(1)} \leq Kp(p-1)t/2$ and $A_t^{(2)} \leq K^2p(p-1)t$, where K is a constant appearing in the Lipschitz condition. Observe also that if for every x, y, z we set

$$H(x, y, z) = [(g^2(x) - g^2(z))(h^2(y) - h^2(z)) + (g^2(y) - g^2(z))(h^2(x) - h^2(z))](y - x),$$

then $H(x, y, z) = (G(x, y) - G(x, z) - G(y, z) + G(z, z))(y - x)$ and

$$\begin{aligned} H(x, y, z) + H(y, z, x) - H(x, z, y) &= 2(z - y)G(y, z) - 2(z - x)G(x, z) \\ &\quad + 2(y - x)G(x, y) + G(x, x)(z - y) - G(y, y)(z - x) + G(z, z)(y - x). \end{aligned}$$

Using the last equality and the fact that $|H(x, y, z)| \leq 2K^2|(y - x)(z - y)(z - x)|$ we can write $2dA^{(3)} = dA^{(4)} + dA^{(5)}$, where $0 \leq A_t^{(4)} \leq K^2p(p-1)(p-2)t/6$ and

$$\begin{aligned} dA_t^{(5)} &= \sum_{i<j<k} \frac{G(\lambda_j, \lambda_j)(\lambda_k - \lambda_i) - G(\lambda_i, \lambda_i)(\lambda_k - \lambda_j) - G(\lambda_k, \lambda_k)(\lambda_j - \lambda_i)}{(\lambda_j - \lambda_i)(\lambda_k - \lambda_i)(\lambda_k - \lambda_j)} dt \\ &= \sum_{i<j<k} \left(\frac{G(\lambda_j, \lambda_j) - G(\lambda_i, \lambda_i)}{\lambda_j - \lambda_i} - \frac{G(\lambda_k, \lambda_k) - G(\lambda_j, \lambda_j)}{\lambda_k - \lambda_j} \right) \frac{1}{\lambda_k - \lambda_i} dt \end{aligned}$$

If $G(x, x) = 2g^2(x)h^2(x)$ is convex then obviously the expression under the last sum and $A^{(5)}$ is non-positive. When $G(x, x)$ is $\mathcal{C}^{1,1}$, (i.e. $|G'(x, x) - G'(y, y)| \leq C|x - y|$) then it is bounded by C and $|A_t^{(5)}| \leq Ct$.

Since finite-variation part of U is finite whenever t is bounded, applying McKean argument, we obtain that U can not explode in finite time with positive probability and consequently $\tau = \infty$ a.s. \square

Remark 1. Note that if $p = 2$ then the assumptions on g^2h^2 can be dropped since in that case $dA^{(3)} \equiv 0$. Observe also that the Proposition 1 holds also in the complex case. The eigenvalues of the process X_t on \mathcal{H}_p verify the system (3.9) and the proof of the Proposition 1 remains valid.

3.4. Matrix Yamada-Watanabe theorem.

Theorem 5. *Consider the matrix SDE (3.1) on \mathcal{S}_p*

$$dX_t = g(X_t)dB_t h(X_t) + h(X_t)dB_t^T g(X_t) + b(X_t)dt$$

where $g, h, b : \mathbf{R} \rightarrow \mathbf{R}$ and $X_0 \in \tilde{\mathcal{S}}_p$. Suppose that

$$|g(x)h(x) - g(y)h(y)|^2 \leq \rho(|x - y|), \quad x, y \in \mathbf{R}, \quad (3.14)$$

where $\rho : (0, \infty) \rightarrow (0, \infty)$ is a Borel function such that $\int_{0+} \rho^{-1}(x)dx = \infty$, that the functions $G(x, y) = g^2(x)h^2(y) + g^2(y)h^2(x)$ is locally Lipschitz and strictly positive on $\{x \neq y\}$ and that b is locally Lipschitz. Then, for $t < \tau$, the pathwise uniqueness holds for the SDE (3.1).

Remark 2. The hypothesis in Theorem 5 on the strict positivity of $G(x, y)$ off the diagonal $\{x = y\}$ is equivalent to the condition that g and h have no more than one zero and their zeros are not common.

Remark 3. In the matrix SDE (3.1) the functions g and h appear only in the martingale part, whereas in the equations (3.2) and (3.3) they intervene also in the finite variation part. That is why a Lipschitz condition on the symmetrized function $g^2 \oplus h^2$ cannot be avoided in a matrix Yamada-Watanabe theorem on \mathcal{S}_p .

Proof. We diagonalize $X_0 = h_0 \lambda_0 h_0^T$. We first show that the equations (3.2) and (3.3) have unique strong solutions when $\Lambda_0 = \lambda_0$ and $H_0 = h_0$. The functions

$$\begin{aligned} b_i(\lambda_1, \dots, \lambda_p) &= b(\lambda_i) + \sum_{k \neq i} \frac{G(\lambda_i, \lambda_k)}{\lambda_i - \lambda_k}, \\ c_{ij}(\lambda_1, \dots, \lambda_p, h_{11}, h_{12}, \dots, h_{pp}) &= \delta_{kj} h_{ik} \frac{\sqrt{G(\lambda_j, \lambda_k)}}{\lambda_j - \lambda_k}, \\ d_{ij}(\lambda_1, \dots, \lambda_p, h_{11}, h_{12}, \dots, h_{pp}) &= -\frac{1}{2} h_{ij} \sum_{k \neq j} \frac{G(\lambda_j, \lambda_k)}{(\lambda_k - \lambda_j)^2} \end{aligned}$$

are locally Lipschitz continuous on $D = \{0 \leq \lambda_1 < \lambda_2 < \dots < \lambda_p\} \times [-1, 1]^r$, $r = p^2$. Thus, they can be extended from the compact sets

$$D_m = \{0 \leq \lambda_1 < \lambda_2 < \dots < \lambda_p < m, \lambda_{i+1} - \lambda_i \geq 1/m\} \times [-1, 1]^r$$

to bounded Lipschitz continuous functions on \mathbf{R}^{p+r} . We will denote by b_i^m , c_{ik}^m and d_{ij}^m such extensions for $m = 1, 2, \dots$.

We consider the following system of SDE (recall that $\beta_{kj} = \beta_{jk}$)

$$\begin{aligned} d\lambda_i^m &= g(\lambda_i^m) d\nu_i + b_i^m(\Lambda^m) dt, \quad i = 1, \dots, p, \\ dh_{ij} &= \sum_{k \neq j} c_{ij}^m(\Lambda^m, H) d\beta_{kj}(t) + d_{ij}^m(\Lambda^m, H) dt, \quad 1 \leq i, j \leq p. \end{aligned}$$

Since $|g(x)h(x) - g(y)h(y)|^2 \leq \rho(|x - y|)$ and $\int_{0+} \rho(x)^{-1} dx = \infty$, by Theorem 2 with $q = \frac{1}{2}p(p-1)$, we obtain that there exists a unique strong solution of the above-given system of SDEs. Using the fact that $D_m \subset D_{m+1}$, $\lim_{m \rightarrow \infty} D_m = D$ and the standard procedure we get that there exists a unique strong solution (Λ_t, H_t) of the systems (3.2) and (3.3) up to the first exit time from the set D . This time is almost surely equal to τ , the first collision time of the eigenvalues.

Suppose that X and \tilde{X} are two solutions of (3.1) with $X_0 = \tilde{X}_0 = h_0 \lambda_0 h_0^T$. Consider the corresponding eigenvalues and eigenvectors processes (Λ, H) and $(\tilde{\Lambda}, \tilde{H})$ with $\Lambda_0 =$

$\tilde{\Lambda}_0 = \lambda_0$ and $H_0 = \tilde{H}_0 = h_0$. We have just proved that $\Lambda(t) = \tilde{\Lambda}(t)$ and $H(t) = \tilde{H}(t)$ for every $t > 0$ a.s., so $X(t) = \tilde{X}(t)$ a.s. \square

The Theorem 5 and the Proposition 1 imply the following global strong existence result for matrix SDEs on the positive cone \mathcal{S}_p^+ .

Corollary 1. *Consider functions $g, h : \mathbf{R}^+ \rightarrow \mathbf{R}^+$. Suppose that b, g^2, h^2 are Lipschitz continuous, $g^2 h^2$ is convex or in class $\mathcal{C}^{1,1}$ and that $G(x, y)$ is strictly positive on $\{x \neq y\}$. Then the matrix SDE (3.1) on \mathcal{S}_p^+ admits a unique strong solution on $[0, \infty)$.*

Proof. Recall that if a non-negative function F is Lipschitz continuous then \sqrt{F} is $1/2$ -Hölder continuous. Observe that if g^2 and h^2 are Lipschitz continuous then $g^2 h^2$ is locally Lipschitz and gh is $1/2$ -Hölder. Thus (3.14) is verified and the Theorem 5 applies on $[0, \tau)$. By the Proposition 1, $\tau = \infty$ almost surely. \square

4. APPLICATIONS

4.1. Noncolliding particle systems of squared Bessel processes. In a recent paper by Katori, Tanemura [14], particle systems of squared Bessel processes $\text{BESQ}^{(\nu)}$, $\nu > -1$, interacting with each other by *long ranged repulsive forces* are studied. If there are N particles, their positions $X_i^{(\nu)}$ are given by the following system of SDEs, see [14] p.593:

$$\begin{aligned} dX_i^{(\nu)}(t) &= 2\sqrt{X_i^{(\nu)}(t)}dB_i(t) + 2(\nu + 1)dt + 4X_i^{(\nu)}(t) \sum_{j \neq i} \frac{1}{X_i^{(\nu)}(t) - X_j^{(\nu)}(t)} \\ &= 2\sqrt{X_i^{(\nu)}(t)}dB_i(t) + 2(\nu + N)dt + 2 \sum_{j \neq i} \frac{X_i^{(\nu)}(t) + X_j^{(\nu)}(t)}{X_i^{(\nu)}(t) - X_j^{(\nu)}(t)}, \quad i = 1, \dots, N \end{aligned}$$

with a collection of independent standard Brownian motions $\{B_i(t), i = 1, \dots, N\}$ and, if $-1 < \nu < 0$, with a reflection wall at the origin. It is shown in [14] and it follows from the Proposition 1 that the collision time $\tau = \infty$ a.s. The Theorem 2 applied to such systems, with a standard use of localization techniques as in the proof of Theorem 5, gives the following

Corollary 2. *The system of SDEs for a particle system of N squared Bessel processes $\text{BESQ}^{(\nu)}$ admits a unique strong solution on $[0, \infty)$.*

4.2. Wishart stochastic differential equations. Wishart processes on \mathcal{S}_p^+ are matrix analogues of squared Bessel processes on \mathbf{R}^+ . Wishart processes with shape parameter n (which corresponds to the dimension of a BESQ on \mathbf{R}^+) are simply constructed as $X_t = N_t^T N_t$ where N_t is an $n \times p$ Brownian matrix. Let $\alpha > 0$ and $B = (B_t)_{t \geq 0}$ be a Brownian $p \times p$ matrix. The Wishart stochastic differential equation for a Wishart process with a shape parameter α is

$$\begin{cases} dX_t &= \sqrt{X_t}dB_t + dB_t^T \sqrt{X_t} + \alpha Idt \\ X_0 &= x_0. \end{cases} \quad (4.1)$$

It was introduced by Bru [3] by first writing the SDE for $X_t = N_t^T N_t$ and next replacing the parameter n by α . It was shown in [3] that if $x_0 \in \tilde{\mathcal{S}}_p$ and $\alpha \in (p-1, p+1)$ then there exists a unique weak solution of (4.1) and the condition $\alpha \geq p+1$ implies that (4.1) has a unique strong solution. We reinforce considerably these results. Our methods apply with $g(x) = \sqrt{x}$, $h(x) = 1$ and $G(x, y) = x + y$ for $x, y \in \mathbf{R}^+$. These functions satisfy the hypotheses of the Theorem 5 and the Proposition 1. First, using the Theorem 3 and the Proposition 1 we obtain

Corollary 3. *For $\alpha > 0$ and $0 \leq \lambda_1(0) < \lambda_2(0) < \dots < \lambda_p(0)$ the eigenvalues $\lambda_i(t)$ never collide, i.e. the first collision time $\tau = \infty$ almost surely.*

Next, the Theorem 5 implies

Corollary 4. *The Wishart SDE (4.1) with $x_0 \in \tilde{\mathcal{S}}_p$ has a unique strong solution on $[0, \infty)$ for $\alpha > 0$.*

Remark 4. Bru [3] showed that for $\alpha > p - 1$ the Wishart processes have the absolutely continuous Wishart laws which are very important in multivariate statistics, see e.g. the monograph of Muirhead [18]. The singular Wishart processes corresponding to $\alpha = 1, \dots, p - 1$ are obtained as $X_t = N_t^T N_t$ where N_t is an $\alpha \times p$ Brownian matrix. Then $X_0 = N_0^T N_0$ has eigenvalue 0 of multiplicity $p - \alpha$ so $x_0 \notin \tilde{\mathcal{S}}_p$.

Thanks to our matrix Yamada-Watanabe Theorem, we discover a new and unexpected class of Wishart processes for $\alpha \in (0, p - 1]$ and $x_0 \in \tilde{\mathcal{S}}_p$, whose study is an interesting research objective.

Remark 5. Let $Q \in \mathcal{S}_p^+$. As observed in Bru [3] p.746-747, X_t is a solution to the matrix SDE (3.1) if and only if $Y_t = QX_tQ$ is a solution of

$$dY_t = \sqrt{Y_t}dB_tQ + QdB_t^T\sqrt{Y_t} + \alpha Q^2dt. \quad (4.2)$$

The process Y_t is a Wishart process with shape parameter α and scale parameter Q^2 . Observe that if $Q \neq \alpha I$, the matrix SDE (4.2) has not the form (3.1) for any $b, h : \mathbf{R}^+ \rightarrow \mathbf{R}^+$. However, using the relation $Y_t = QX_tQ$ and Theorem 5 it is easy to see that the SDE (4.2) has a unique strong solution.

Another perturbation of the Wishart SDE (4.1) is the equation for the Wishart process with constant drift $c > 0$, which may be also viewed as a squared matrix Ornstein-Uhlenbeck process

$$dX_t = \sqrt{X_t}dB_t + dB_t^T\sqrt{X_t} + \alpha Idt + cX_tdt, \quad X_0 \in \tilde{\mathcal{S}}_p. \quad (4.3)$$

This equation has the form (3.1) with $g(x) = \sqrt{x}$, $h(x) = 1$ and $b(x) = cx$. By Theorem 5 and Proposition 1, it has a unique strong solution with $t \in [0, \infty)$ for any $\alpha, c > 0$. More general squared matrix Ornstein-Uhlenbeck processes were first studied by Bru [3] and recently by Mayerhofer et al. [17]. Our strong existence and uniqueness result for (4.3) is not covered by these papers.

4.3. Matrix Jacobi processes. Let 0_p and I_p be zero and identity $p \times p$ matrices. Define $\mathcal{S}_p[0, I] = \{X \in \mathcal{S}_p \mid 0_p \leq X \leq I_p\}$. Denote by $\hat{\mathcal{S}}_p[0, I] = \{X \in \mathcal{S}_p \mid 0_p < X < I_p\}$ and by $\tilde{\mathcal{S}}_p[0, I]$ the set of matrices in $\mathcal{S}_p[0, I]$ with distinct eigenvalues. A matrix Jacobi process of dimensions (q, r) , with $q \wedge r > p - 1$, and with values in $\mathcal{S}_p[0, I]$, was defined and studied by Doumerc [10] as a solution of the following matrix SDE, with respect to a $p \times p$ Brownian matrix B_t

$$\begin{cases} dX_t &= \sqrt{X_t}dB_t\sqrt{I_p - X_t} + \sqrt{I_p - X_t}dB_t^T\sqrt{X_t} + (qI_m - (q + r)X_t)dt \\ X_0 &= x_0 \in \mathcal{S}_p[0, I]. \end{cases} \quad (4.4)$$

In [10] Th.9.3.1, p.135 it was shown that if $q \wedge r \geq p + 1$ and $x_0 \in \hat{\mathcal{S}}_p[0, I]$ then (4.4) has a unique strong solution in $\hat{\mathcal{S}}_p[0, I]$. In the case q or $r \in (p - 1, p + 1)$ and $x_0 \in \tilde{\mathcal{S}}_p[0, I]$ the existence of a unique solution in law was proved in [10]. Our methods allow one to strengthen the results of Doumerc.

Corollary 5. *Let $q \wedge r > p - 1$ and $x_0 \in \tilde{\mathcal{S}}_p[0, I]$. Then the matrix SDE (4.4) has a unique strong solution for $t \in [0, \infty)$.*

Proof. We apply Theorem 5 and Proposition 1 with $g(x) = \sqrt{x}$, $h(x) = \sqrt{1-x}$ and $b(x) = q - (q+r)x$. \square

4.4. β -Wishart and β -Jacobi processes. As known from [1] and by Theorem 3, the eigenvalues of the Wishart process verify the following system of SDEs

$$d\lambda_i = 2\sqrt{\lambda_i}d\nu_i + \left(\alpha + \sum_{k \neq i} \frac{\lambda_i + \lambda_k}{\lambda_i - \lambda_k} \right) dt.$$

Let $\beta > 0$. One calls a β -Wishart process a solution of the system of SDEs

$$d\lambda_i = 2\sqrt{\lambda_i}d\nu_i + \beta \left(\alpha + \sum_{k \neq i} \frac{\lambda_i + \lambda_k}{\lambda_i - \lambda_k} \right) dt. \quad (4.5)$$

The β -Wishart processes were studied by Demni [7]. In the theory of random matrices and its physical applications, the β -Wishart processes are related to Chiral Gaussian Ensembles, which were introduced as effective (approximation) theoretical models describing energy spectra of quantum particle systems in high energy physics. Usually a symmetry of Hamiltonian is imposed and it fixes the value of β to be 1, 2 or 4, respectively in real symmetric, hermitian and symplectic cases. On the other hand, from the point of view of statistical physics, β is regarded as the inverse temperature, $\beta = 1/(k_B T)$, and should be treated as a continuous positive parameter. In this sense, the β -Wishart systems are statistical mechanics models of “log-gases” (The strength of the force between particles is proportional to the inverse of distances. Then the potential, which is obtained by integrating the force, is logarithmic function of the distance. So the system is called a “log-gas”). For more information on log-gases, see the recent monograph of Forrester [11].

In [7] the existence and uniqueness of strong solutions of the SDE system (4.5) was established for $\beta > 0$, $\alpha > p - 1 + \frac{1}{\beta}$ and $t \in [0, \tau \wedge R_0)$, where $R_0 = \inf\{t \mid \lambda_1 = 0\}$. Our Theorem 2 and Proposition 1 imply the following

Corollary 6. *The SDE system (4.5) with $0 \leq \lambda_1(0) < \lambda_2(0) < \dots < \lambda_p(0)$ has a unique strong solution for $t \in [0, \infty)$, for any $\alpha > 0$ and $\beta > 0$.*

The β -Jacobi processes $(\lambda_i), i = 1, \dots, p$ are $[0, 1]^p$ -valued processes generalizing processes of eigenvalues of matrix Jacobi processes defined by (4.4):

$$d\lambda_i = 2\sqrt{\lambda_i(1-\lambda_i)}d\nu_i + \beta \left(q - (q+r)\lambda_i + \sum_{k \neq i} \frac{\lambda_i(1-\lambda_k) + \lambda_k(1-\lambda_i)}{\lambda_i - \lambda_k} \right) dt. \quad (4.6)$$

Indeed, for $\beta = 1$ the formula (4.6) was shown in [10] and it follows directly from the Theorem 3. β -Jacobi processes were recently studied by Demni in [8]. He showed that the system (4.6) has a unique strong solution for all time t when $\beta > 0$ and $q \wedge r > m - 1 + 1/\beta$. As an application of Theorem 2 and Proposition 1 we strengthen essentially this result.

Corollary 7. *The SDE system (4.6) with $0 \leq \lambda_1(0) < \lambda_2(0) < \dots < \lambda_p(0) \leq 1$ has a unique strong solution for $t \in [0, \infty)$, for any $\beta > 0$ and $q \wedge r > m - 1$.*

Remark 6. It would be interesting to extend our generalization of the Yamada theorem to the SDE's considered by Cépa-Lépingle [4]. On the other hand the Jacobi eigenvalues processes being an important example of the radial Cherednik processes, we conjecture that the strong existence and unicity would hold for radial Cherednik processes. For radial Dunkl processes this is proved by Demni [6], using [4].

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